# APPROXIMATION OF INDEFINITE INTEGRAL WITH SINGULARITY USING VARIABLE TRANSFORMATION METHOD

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**Abstract**

A numerical scheme based on Sinc function was applied to an indefinite integral with possible endpoint singularities in the integrand. The scheme employed Sinc spaces of approximation and a variable transformation function constructed with a composition of trigonometric functions. With this method, the singularity in the integrand is moved to infinity allowing a possible evaluation of the integral without consideration of the endpoint singularity. The procedure was demonstrated with numerical examples to justify its suitability for numerical integration over the prescribed interval of integration. The error bound in the numerical results satisfies the established convergence rates associated with Sinc methods with single exponential decay with increase number of evaluation points N. Finally, the results shown on tables and graphs validate the efficiency of the numerical scheme.

**Keywords:** Indefinite integral; Endpoint singularity; Composite trigonometric function; Sinc approximation; Variable transformation function

## 1.0 Introduction

Numerical methods are especially useful in situations where analytical evaluation is very difficult to obtain or the result is too unwieldy to permit further processing for meaningful interpretation. The widespread availability of fast and efficient computers has renewed interest in the development of exceptionally accurate numerical methods for handling realistic problems.

One area that this interest is employed is in the integration of functions in which the integrand may have endpoint singularities. This challenge can be overcome by the use of variable transformation, a transformation that relates the original (finite) interval of integration to the real line $\mathbb{R}$ in such a way that the singularities are moved to infinity.

Early contributors to this discourse include Haber (1993) and Kearfott (1983) whose work employed the tanh transformation in the strip region. Mohammad and Mori (2003) further improved the results with use of double exponential formula. Recently, the application of error functions has been employed by John & George (2024) as a variable transformation function in the approximate solution of indefinite integral.

The extension of such procedure with composite trigonometric functions is the motivation in this research work.

The paper considers the application of the variable transformation, John et al (2024)

$$x = \varphi(t) = \frac{b-a}{2} sin\left(tan^{-1}(t)\right) + \frac{b+a}{2}, t \in (-\infty, \infty) \tag{1}$$

to the integral

$$\int_0^s f(x)dx, 0 < s < 1 \tag{2}$$

where the integrand $f(x)$in (2) is assumed to be analytic in $x \in (0, 1)$ except at $x = 0$ or $x = 1$ where singularity may occur. Equation (1) is a composition of trigonometric functions with the conditions.

$\varphi(-\infty) = 0, \varphi(0) = 0.5, \varphi(\infty) = 1.$

The inverse and the derivative of (1) are

$$t = \varphi^{-1} = (tan\,(sin^{-1}\,(2x\text{-}1)))$$

and

$$\varphi^{'}(t) = \frac{\frac{1}{2}\cos\left(tan^{-1}(t)\right)}{1+t^2} \qquad (4)$$

respectively.

## 2.0 Materials and Methods

In this section the available tools for the implementation of the variable transformation method for the numerical integration of (2).

## 2.1 Variable transformation Technique

Given a variable transformation function φ *(t)*, the integral (2) becomes

$$\int_0^s f(x)dx = \int_{-\infty}^{\beta} u(t)dt \quad u(t) = f(\varphi(t))\varphi^{'}(t),\ \beta = \varphi^{-1}(s). \qquad (5)$$

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Following Mori & Mohammad (2003) and from (5), it is assumed that f(x) satisfies

$$f(x) = \begin{cases} O((1+x)^{\alpha-1})as\ x \to -1 \\ O((1-x)^{\alpha-1})\ as\ x \to 1 \end{cases} \text{for } \alpha > 0 \qquad (6)$$

such that for a single exponential transformation

$$f(\varphi(t))\varphi^{'}(t) = O(\exp\,(-2(\alpha-\epsilon)|t|))\text{as } t \to \pm\infty \qquad (7)$$

where $\alpha$ is a constant and $0 < \in < 1$,

## 2.2 The Sinc Function

The function

$$S(j,h)(t) = S\left(\frac{t}{h} - j\right) = \frac{sin\pi(\frac{t}{h} - j)}{\pi(\frac{t}{h} - j)}\ , j = 0, \pm 1, \pm 2, \ldots \qquad (8)$$

is known as shifted Sinc function Stenger (1993), and for $t_k = kh$,

$$S(j,h)(kh) = \begin{cases} 0, k \neq j \\ 1, k = j \end{cases}.$$

## 2.3 Trapezoidal Rule

Given the integral

$$I = \int_{-\infty}^{\infty} F(u)du \tag{9}$$

The trapezoidal rule approximation to (3) is given by

$$T_h = h \sum_{j=-\infty}^{\infty} F(jh), h > 0 \tag{10}$$

Consequently, for t e function F(u) defined on the whole real line, $(-\infty, \infty)$ as

$$F(u) \approx \sum_{j=-N}^{N} F(jh)S(j,h)(u), \quad u \in \mathbb{R} \tag{11}$$

thus

$$\int_{-\infty}^{\infty} F(u)du \approx \sum_{j=-N}^{N} F(jh) \int_{-\infty}^{\infty} S(j,h)(u)du = h \sum_{j=-N}^{N} F(jh) \tag{13}$$

From (5), (8) and (13),

$$\int_{-\infty}^{\beta} u(t)dt \;\; u(t) = h \sum_{-\infty}^{\infty} u\,(jh) \left( \frac{1}{2} + \frac{1}{\pi} Si \left( \frac{\pi\varphi^{-1}(s)}{h} - j\pi \right) \right) + Ohe^{-\frac{\pi d}{h}} \tag{14}$$

where

$$Si\,(x) = \int_0^{\tau} \frac{\sin\tau}{\tau}\, d\tau\,. \tag{15}$$

## 2.4 Convergence Theorems

**Definition 2.1: Okayama et al (2011)**

Let α be a positive constant, and let $D$ be a bounded and simply-connected domain which satisfies $(a, b) \subset D$. Then $La\,(D)$ denotes the family of functions $f$ that satisfy the following conditions: (i) $f$ is analytic in $D$; (ii) there exists a constant $C$ such that for all $z$ in $D$

$$|f(z)| \leq C|Q(z)|^{\alpha} \tag{16}$$

where the function $Q$ is defined by

$$Q(z) = (z - a)(b - z).$$

For the implementation of the single exponential transformation in the above Definition 2.1, the domain $D$ considered to be the region $\phi\,(D_d) = \{z = \phi\,(\mu) : \mu \in D_d\}$ where $D_d = \{z \in \mathbb{C}: |im\ \mu| < d\}$ denotes a strip region of width $2d$. This gives the image of the region under consideration as,

$$\varphi(\mathcal{D}_d) = \{z \in \mathbb{C}: |arg(\tan(\sin^{-1}(2x-1)))| < d\} \tag{17}$$

**Theorem 1** Stenger (1993) and Stenger (2011)

Let $f \in L_\alpha\varphi\ (Dd)$ for $d$ with $0 < d < \pi$, let $N$ be a positive integer and let $h$ be selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}} \tag{18}$$

then there is a constant $C$ independent of

$$\max_{a \le x \le b} \left| f(x) - \sum_{j=-N}^{N} f(\varphi(jh)S(j,h)(\{\varphi\}^{-1}(x)) \right| \le C\sqrt{N}e^{-\sqrt{\pi d\alpha N}}\,. \tag{19}$$

The choice of $h$ is optimal and satisfies (20) based on Sugihara (2002).

**Theorem 2:** Kearfott (1983)

Let $(fQ) \in L_\alpha\varphi\ (Dd)$ for $d$ with $0 < d < \pi$, let $N$ be a positive integer and let $h$ be selected by the formula (18), then there is a constant $C$ independent of $N$, such that

$$\left| \int_a^s f(t)dt - h\sum_{j=-N}^{N} f(\varphi(jh)S(j,h)(\{\varphi\}'(jh))J(j,h) \right| \le CN^2 e^{-\sqrt{\pi d\alpha N}} \tag{20}$$

where

$$J(j,h) = \left(\frac{1}{2} + \frac{1}{\pi}Si\left(\frac{\pi\varphi^{-1}(s)}{h} - j\pi\right)\right).$$

From Theorem 1 and Theorem 2, equation (2) has the approximation,

$$\int_0^s f(x)dx = h\sum_{-\infty}^{\infty} f(\varphi(jh))\varphi'(jh)\left(\frac{1}{2} + \frac{1}{\pi}Si\left(\frac{\pi\phi^{-1}(s)}{h} - j\pi\right)\right) + O\left(N^{\frac{1}{2}}\exp\left(-\sqrt{\pi dN}\right)\right). \tag{21}$$

## 3.0 Results

In this section, the formula (1) is applied to equation (2) using the following examples:

The maximum absolute error between the exact solution and the approximate solution at Sinc points $x_k$ is defined by $|E_N(h(\varphi))|$ with respect to $L_\infty$ norm such that:

$$|E_N(h(\varphi))| = \max_{k=-N-1,-N,\ldots N,N+1} |\text{Exact} - \text{Approx}|. \quad (22)$$

Numerical computations were carried out with the help of MATLAB R12 where $h$ was chosen as in (18)

**Example 3.1**

$$\int_0^s \frac{1}{\pi\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\arcsin s.$$

Example 3.1 is a modification of Haber (1993) with the integrand exhibiting endpoint singularity at 1. For this example, $d = \pi/2$ and $a = 1$.

**Example 3.2**

$$\int_0^s \frac{\log 2x}{1-x}\,dx = \frac{s\log 2s}{(1-s)+\log(1-s)}$$

The integrand in Example 3.2 exhibits endpoint singularity at $s = 0$ and $s = 1$.

**Table 1: Maximum error and error ratio for Example 3.1**

| N | H | $\|E_N(h(\varphi))\|$ | Error Ratio |
|---|---|---|---|
| 10 | 0.7025 | $1.73 \times 10^{-2}$ | - |
| 20 | 0.4967 | $8.6 \times 10^{-3}$ | 2.01 |
| 30 | 0.4056 | $4.4 \times 10^{-3}$ | 1.95 |
| 40 | 0.3512 | $1.9 \times 10^{-3}$ | 2.32 |
| 50 | 0.3142 | $3.1704 \times 10^{-4}$ | 5.99 |

**Table 2: Maximum error and error ratio for Example 3.2**

| N | H | $\|E_N(h(\varphi))\|$ | Error Ratio |
|---|---|---|---|
| 10 | 0.7025 | $3.85 \times 10^{-2}$ | - |
| 20 | 0.4967 | $1.67 \times 10^{-2}$ | 2.31 |
| 30 | 0.4056 | $1.13 \times 10^{-2}$ | 1.48 |
| 40 | 0.3512 | $8.7 \times 10^{-3}$ | 1.30 |
| 50 | 0.3142 | $7.2 \times 10^{-3}$ | 1.21 |

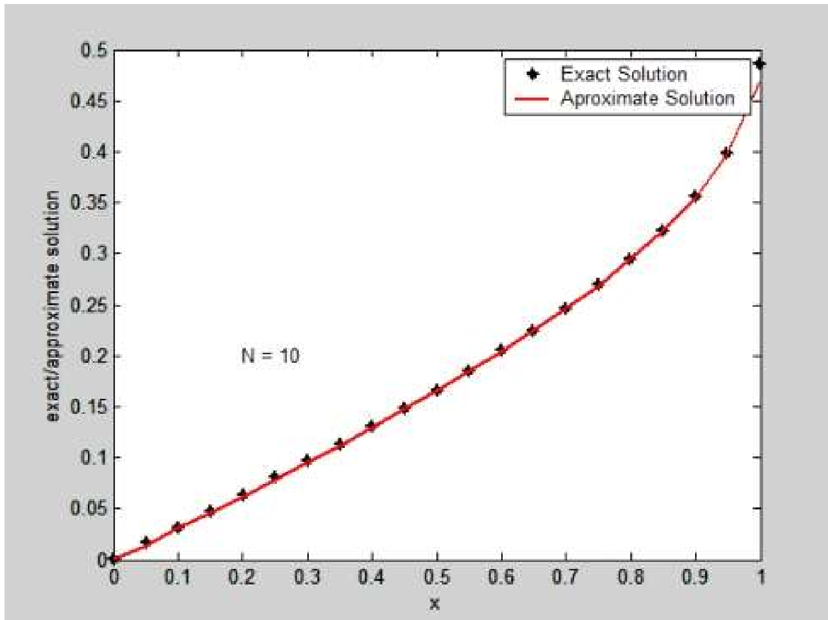


Fig.1: Exact and approximate solution Example 3.1 at $N = 10$

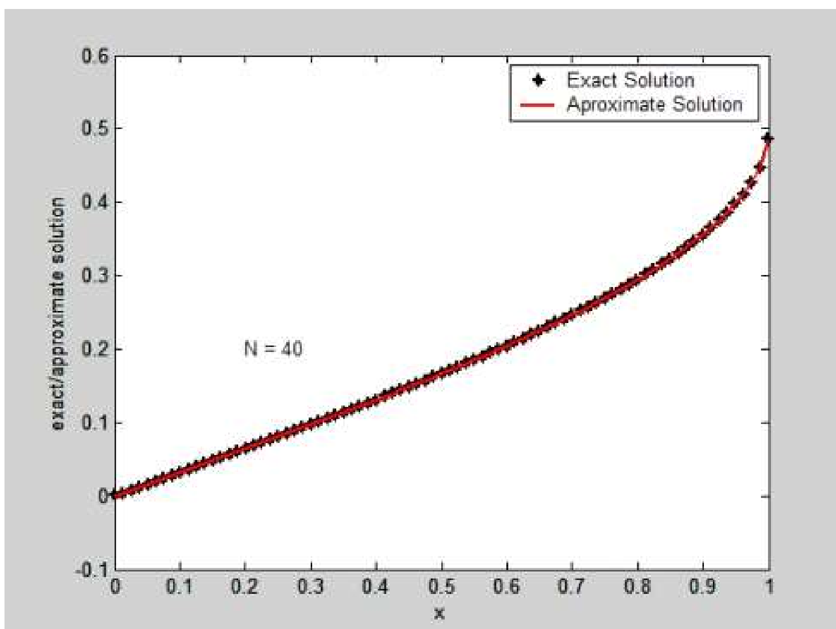


Fig.2: Exact and approximate solution Example 3.1 at $N = 40$

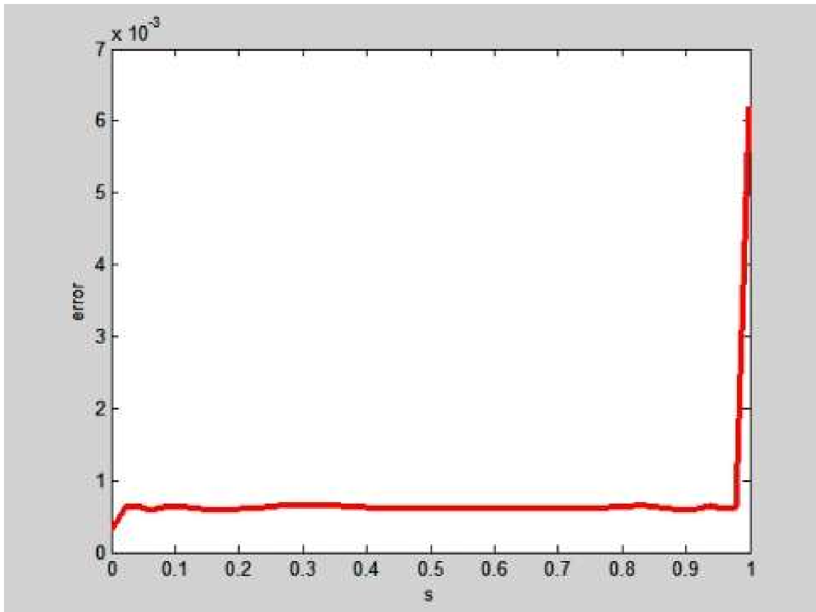


Fig.3: Error against - for Example 3.1 at $N = 25$

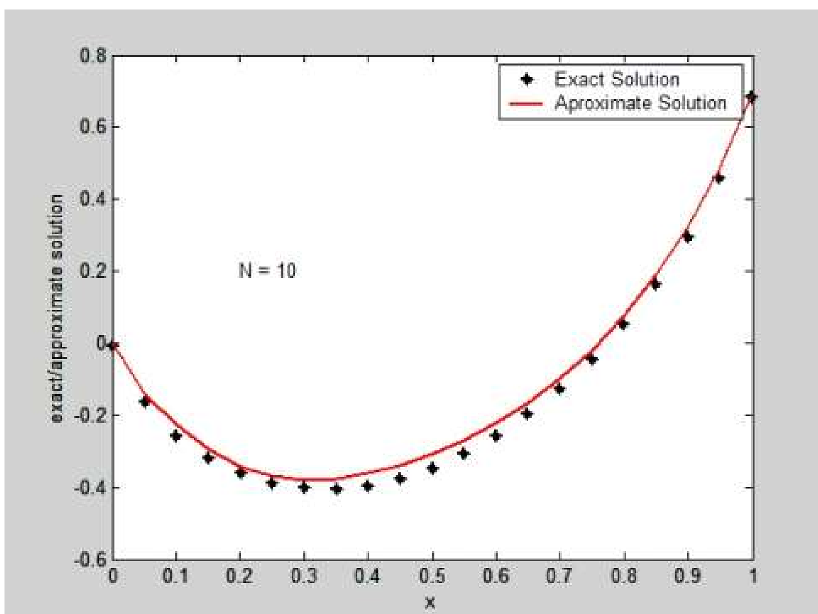


Fig.4: Exact and approximate solution Example 3.2 at $N = 10$

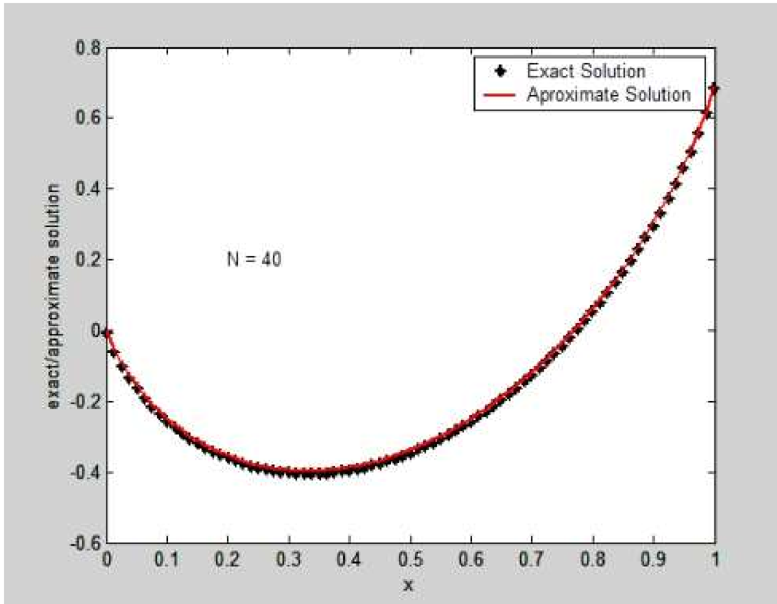


Fig. 5: Exact and approximate solution Example 3.2 at $N = 40$

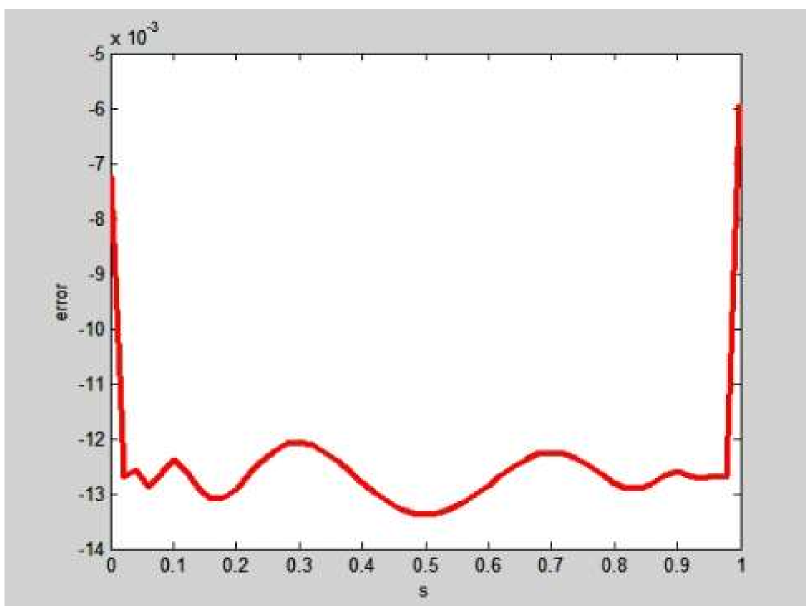


Fig. 4: Error against - for Example 3.2 at $N = 25$

## 4.0 Discussion

The focus of this paper was to demonstrate the application of composite trigonometric function for the numerical approximation of an indefinite integral in the region *(0, s)*, where the integrand appears to be singular in one or both endpoints. The properties of the variable transformation function is clearly defined alongside the parameters required for a successful implementation of the method.

This article employed procedures outlined in some already established research works on single exponential Sinc approximations for numerical integration; such as the theoretical result for optimal parameters to guide in the choice of the step-size of the numerical scheme and the determination of the convergence rate of the scheme.

The results of the numerical examples are shown in Tables 1 and Table 2 for Examples 3.1 and 3.2 respectively. These results demonstrate the error decay over the interval of integration with the increase number of evaluation (N). In Figures 1 and 2 the plots show the comparison between the exact and approximate solutions for N = 10 and N = 40 respectively for Example 3.1 and Figures 4 and 5 show the comparison between the exact and approximate solutions for N = 10 and N = 40 respectively for Example 3.2 respectively. The improvement of the accuracy is seen with respect to the increase of number of evaluations N. The behavior of the error over the interval for Example 3.1 and Example 3.2 for N = 25 are illustrated in Figure 3 and Figure 6 respectively.

In Figure 3 and Figure 6, the horizontal ranges are $0 < s < 1$ respectively. The vertical range for Figure 3 ranges from $-10^{-3}$ to $10^{-3}$ and that of Figure 6 ranges from $-10^{-3}$ to $10^{-3}$

## 5.0 Conclusion

In this work, the numerical indefinite integration was implemented using a composite trigonometric function as a variable transformation formula with the help of Sinc function. The efficiency of the method as demonstrated with numerical examples agrees with the theoretical convergence results.

## 6.0 Recommendations

The method demonstrated opens up a new frontier for researchers with interest in numerical analysis and approximation theory. Thus, the following recommendations will help interested researchers to further contribute to this discourse.

- The development trigonometric functions as alternatives to hyperbolic functions for the construction of variable transformation functions.
- The method employed is based on a single exponential formula, further studies should consider developing a double exponential formula by improving the current for improved convergence.

## 7.0 Recommendation for further studies

Since the topic is topical, expansive and daring, there is a compelling need for extensive studies. Subsequent work will seek to extend the application of the method to other intervals, while considering alternative approaches to convergence of the method with respect to the determination of optimal parameters. Also, factors required for improved convergence will be studied. An extension of the procedure for the solution of weakly singular integral equations will also be considered.

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